SOME NON-PARAMETRIC TESTS OF WHETHER THE LARGEST OBSERVATIONS OF A SET ARE TOO LARGE OR TOO SMALL

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SOME NON-PARAMETRIC TESTS OF WHETHER THE LARGEST OBSERVATIONS OF A SET ARE TOO LARGE OR TOO SMALL

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1. Summary. ~~Let us~~ consider a large number n of observations which are statistically independent and drawn from continuous symmetrical populations. This paper presents some non-parametric tests of whether the r largest observations of the set are too large to be consistent with the hypothesis that these populations have a common median value. Tests of whether the r largest observations are too small to be consistent with this hypothesis are also considered. Here r is a given integer which is independent of n.

Subject to some weak restrictions, it is shown that the significance level of a test of the type presented tends to a value α as n increases. For no admissible value of n, however, does the significance level of this test exceed 2α. If whether the largest observations are too large is considered, tests with values of α suitable for significance levels can be obtained for $r \geq 4$. Values of α suitable for significance levels can be obtained for any value of r if whether the largest observations are too small is investigated (n large).

Properties of the power functions of these tests are considered for the special case in which the r largest observations are from populations with common median θ, the remaining observations are from populations with common median ϕ, and each population has the property that the distribution of the quantity

$$\text{(sample value)} - \text{(population median)}$$

is independent of the value of the population median. For tests of $\theta > \phi$, the power function tends to zero as $\theta - \phi \longrightarrow -\infty$ and to unity as $\theta - \phi \longrightarrow \infty$. For tests of $\phi > \theta$, the power function tends to unity as $\theta - \phi \longrightarrow -\infty$ and to zero as $\theta - \phi \longrightarrow \infty$.

Analogous tests of whether the smallest observations of a set are too small or too large can be obtained from the tests of the largest observations by symmetry considerations.

If there is strong reason to believe that the set of observations is a random sample from a continuous population, the tests presented in this paper can be used to decide whether the population is symmetrical. Tests of this nature are sensitive to symmetry in the tails of the population but not to symmetry in the central part.

2. Introduction and statement of tests. The tests derived in this paper are applicable to situations of the following two types:

(a). It is known that the observations are independent and from continuous symmetrical populations (i.e., each population has a continuous cdf $F(x)$ such that $F(x - \phi) = 1 - F(\phi - x)$, where ϕ is the population median). It is desired to test whether the largest few observations are too large (or too small) to be consistent with the assumption that the populations have a common median value (if the 50% point of a continuous symmetrical population is not unique, the median of this population is defined to be the midpoint of the interval of 50% points).

(b). It is known that the observations are independent and from continuous populations with a common median value (e.g., the observations may be a sample from a continuous population). It is desired to test whether these populations are symmetrical (with emphasis on the tails of the population).

With respect to (a), perhaps the most common practical application is that where the observations are assumed to be a sample from a continuous symmetrical population of some special type (e.g., normal) but the values of the largest few observations make this assumption questionable. The non-parametric tests presented for (a) are easily applied and a significant result for a non-parametric test automatically implies that the observations are not a sample from the specified type of population. Furthermore, if a parametric test of this situation (i.e., a test based on the assumption of a sample from this special type of population) is significant, the non-parametric tests are useful in determining whether it is possible that the observations might be a sample from a continuous symmetrical population of some other type.

With respect to (b), perhaps the most common application is that where the set of observations can be considered to be a sample from a continuous population and it is desired to test whether this population is symmetrical in the tails.

Now let us consider the forms of the tests. Let $x(1),\cdots, x(n)$ represent the values of the n observations arranged in increasing order of magnitude. Then $x(n+1-r), x(n+2-r),\cdots, x(n)$ are the r largest observations of the set. For situations of type (a), the tests of whether the r largest observations are too large are of the form

(1) <u>Accept that the r largest observations are too large to be consistent with the hypothesis that the populations have a common median if</u>

$$\min\left[x(n+1-i_k) + x(j_k);\ 1 \le k \le s \le r\right] > 2x(W_\alpha),$$

where the i's, j's and r are integers such that

$$i_s = r,\ i_u < i_{u+1},\quad j_v < j_{v+1},\quad j_s < W_\alpha < n+1-r,$$

α is defined by

$$\alpha = \Pr\left\{\min\left[x(n+1-i_k) + x(j_k)\right] > 2\phi \,\middle|\, \phi = \text{common median}\right\},$$

and $W_\alpha = W_\alpha(n)$ is the smallest integer satisfying the relation

$$\Pr\left[x(W_\alpha) < \phi \,\middle|\, \phi = \text{common median}\right] \le \alpha. \tag{1'}$$

In testing the hypothesis of (1), the principle followed is to choose $x(n+1-r)$ and some subset of $x(n+2-r),\cdots, x(n)$ for use in the test. The integer s represents the total number of order statistics selected from $x(n+1-r),\cdots, x(n)$.

The value of $\alpha = \alpha(i_1,\cdots,i_s;\ j_1,\cdots,j_s)$ is independent of n and is given by equation (5) in section 3. Table 1 contains some values of the i's, j's and s which yield values of α suitable for significance levels. For test (1), values of α suitable for significance levels can be obtained for $r \ge 4$.

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If the n independent observations satisfy the additional conditions

(A)

(i). Asymptotically ($n \to \infty$), $x(W_\alpha)$ is statistically independent of $\min[x(n+1-i_k) + x(j_k);\ 1 \le k \le s]$.

(ii). The standard deviations of $x(W_\alpha)$ and $\min[x(n+1-i_k) + x(j_k);\ 1 \le k \le s]$ exist for all $n \ge i_s + j_s - 1$ and the limiting ratio ($n \to \infty$) of these standard deviations is either zero or infinite.

(iii). Let the notation $\sigma(z)$ denote the standard deviation of z. Then, if the populations have a common median ϕ, asymptotically the cdf's of $[x(W_\alpha) - \phi]/\sigma[x(W_\alpha)]$ and $\{\min[x(n+1-i_k) + x(j_k)] - 2\phi\}/\sigma\{\min[x(n+1-i_k) + x(j_k)]\}$ are continuous at the point zero.

then the significance level of test (1) approaches the value α as n tends to infinity.

Although conditions (A) may appear to be complicated, they are not very restrictive. These conditions are satisfied if the n observations are a sample from a continuous population of the type usually encountered in practical situations (i.e., approximated in practical situations). Perhaps the most well known type of continuous symmetrical population for which a sample does not satisfy conditions (A) is that with a triangular probability density function Part (ii) of conditions (A) is not satisfied for a sample from a population of this type.

For large n, relation (1') with the equality sign is approximately satisfied if $W_\alpha = \frac{1}{2}n + \frac{1}{2}K_\alpha\sqrt{n}$, (i.e., the largest integer contained in $\frac{1}{2}n + \frac{1}{2}K_\alpha\sqrt{n}$). Here K_α is the standardized normal deviate exceeded with probability α. This value for W_α was obtained from the normal approximation to the binomial theorem and furnishes a reasonably accurate solution of (1') with the equality sign for $n > 10$, (see [1]).

As an example of a test of type (1), let $r = 5$, $s = 2$, $j_1 = 1$, $j_2 = 2$, $i_1 = 4$, $i_2 = 5$. Then $\alpha \doteq .0547$ and the test is (approximately)

<u>Accept the specified alternative of (1) if</u>

$$\min[x(n-3)+x(1),\ x(n-4)+x(2)] > 2x\left(\frac{1}{2}n + \frac{1}{2}K_{.0547}\sqrt{n}\right).$$

That this is a test of whether the 5 largest observations are too large is intuitively evident from the fact that a significant result will be obtained only if both

$$x(n-3) > 2x\left(\frac{1}{2}n + \frac{1}{2}K_{.0547}\sqrt{n}\right) - x(1) \tag{2}$$

$$x(n-4) > 2x\left(\frac{1}{2}n + \frac{1}{2}K_{.0547}\sqrt{n}\right) - x(2).$$

If the smallest two of the five largest observations are too large, it seems reasonable to suppose that all of the five are too large. A similar interpretation exists for all tests of type (1).

The type (a) tests of whether the largest observations are too small are of the form

(3)

<u>Accept that the r largest observations are too small to be consistent with the hypothesis that the populations have a common median value if</u>

$$\max[x(n+1-j_k) + x(i_k);\ 1 \le k \le s \le r] < 2x(n+1-W_\alpha),$$

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where $j_s = r$, $j_v < j_{v+1}$, $i_u < i_{u+1}$, $i_s < n+1-W_\alpha < n+1-r$, and both α and W_α are defined in (1). From the results for test (1) and symmetry considerations, the significance level of test (3) tends to α as $n \longrightarrow \infty$ if conditions (A) are satisfied; it does not exceed 2α for any admissible value of n. For test (3), values of α suitable for significance levels can be obtained for all values of r (n sufficiently large).

As indicated by (2), the tests of whether the largest observations are too large can also be interpreted as tests of whether the smallest observations are too large. Similarly the tests of whether the largest observations are too small can also be interpreted as tests of whether the smallest observations are too small.

The above discussion presents intuitive reasons for believing that tests (1) and (3) are suitable for the situations to which they are applied. To obtain a semi-quantitative measure of the suitability of these tests, this paper investigates the special case in which the r largest observations are from continuous symmetrical populations with common median θ, the remaining observations are from continuous symmetrical populations with common median ϕ, and each population has the property that the distribution of $x - \phi$ is independent of ϕ, where x is an observation from the population and ϕ is the median of the population. The power function of a test of type (1) or (3) is defined to be the probability that the test is significant given the value of $\theta - \phi$. It is found that the power functions of these tests have several desirable properties: For test (1), the power function tends to zero as $\theta - \phi \longrightarrow -\infty$, is a monotonically increasing function of $\theta - \phi$ for $\theta - \phi < 0$, and tends to unity as $\theta - \phi \longrightarrow \infty$. For test (3), the power function tends to zero as $\theta - \phi \longrightarrow \infty$, is monotonically decreasing for $\theta - \phi < 0$, and tends to unity as $\theta - \phi \longrightarrow -\infty$.

For testing whether the populations are symmetrical in the tails given that they are continuous and have a common median, i.e., situation (b), a combination of (1) and (3) is used. The resulting test is

<u>Accept that the populations are not symmetrical</u>

<u>in the tails if either</u>

$$\min\left[x(n+1-i_k) + x(j_k);\ 1 \le k \le s\right] > 2x(W_\alpha)$$

<u>or</u>

$$\max\left[x(n+1-j_k) + x(i_k);\ 1 \le k \le s\right] < 2x(n+1-W_\alpha),$$

where $\alpha < \frac{1}{2}$, $i_u < i_{u+1}$, $j_v < j_{v+1}$, $j_w \le i_w$, $j_s < W_\alpha < n+1-i_s$, and both α and W_α are defined in (1). Since both inequalities in (4) can not be satisfied simultaneously, the significance level of test (4) tends to 2α as $n \longrightarrow \infty$ if conditions (A) are satisfied; it never exceeds 4α for any admissible value of n.

The asymptotic distribution $(n \longrightarrow \infty)$ of $x(W_\alpha)$ is usually not very sensitive to symmetry of the populations. For example, if the n observations are a sample from a population with a probability density function $f(x)$ such that $f(\phi) \ne 0$, (ϕ = population 50% point), and $f'(x)$ exists and is continuous in a neighborhood of $x = \phi$, it can be shown that the only property of $f(x)$ which influences the asymptotic distribution of $x(W_\alpha)$ is the value of $f(\phi)$. Thus, since a type (1) test investigates both whether the largest observations are too large and

TABLE 1

SOME VALUES OF α FOR $s \leq 5$

α	s	i_1	i_2	i_3	i_4	i_5	j_1	j_2	j_3	j_4	j_5
.0625	1	4					1				
.0312	1	5					1				
.0156	1	6					1				
.0078	1	7					1				
.0039	1	8					1				
.0352	1	7					2				
.0195	1	8					2				
.0107	1	9					2				
.0469	2	4	5				1	2			
.0234	2	5	6				1	2			
.0117	2	6	7				1	2			
.0059	2	7	8				1	2			
.0391	3	4	5	6			1	2	3		
.0195	3	5	6	7			1	2	3		
.0098	3	6	7	8			1	2	3		
.0459	4	4	5	6	7		1	2	3	4	
.0229	4	5	6	7	8		1	2	3	4	
.0115	4	6	7	8	9		1	2	3	4	
.0308	5	4	5	6	7	8	1	2	3	4	5
.0154	5	5	6	7	8	9	1	2	3	4	5
.0077	5	6	7	8	9	10	1	2	3	4	5

whether the smallest observations are too large (to be consistent with the assumption of symmetry), while a type (3) test investigates both whether the largest observations are too small and whether the smallest observations are too small, test (4) should be suitable for testing whether a population has symmetrical tails.

3. Theorems and derivations. The fundamental fact used in this paper is that, if the observations are from continuous symmetrical populations with common median ϕ, the value of

$$\alpha = \Pr\left\{\min[x(n+1-i_k) + x(j_k);\ 1 \le k \le s] > 2\phi\right\}$$
$$= \Pr\left\{\max[x(n+1-j_k) + x(i_k);\ 1 \le k \le s] < 2\phi\right\}$$

is independent of n for the values of n permitted in the tests. This result is a special case of the following theorem

Theorem 1. Consider a set of n independent observations from continuous symmetrical populations with common median ϕ. Let $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$ be fixed sets of integers whose values are independent of n. Then the value of

$$\Pr\left\{\beta\text{th largest of } [x(n+1-j_k) + x(i_k);\ 1 \le k \le s] < 2\phi\right\}$$

is the same for all values of n which are $\ge i_s + j_s - 1$. In particular

$$(5)\qquad \alpha = 2^{-w}\left\{1 + m(1) + \sum_{h_1=1}^{m(2)}[m(1) - h_1] + \sum_{h_2=1}^{m(3)}\sum_{h_1=1}^{m(2)-h_2}[m(1) - h_1 - h_2] + \cdots \right.$$
$$\left. + \sum_{h_{u-1}=1}^{m(u)}\sum_{h_{u-2}=1}^{m(u-1)-h_{u-1}}\cdots\sum_{h_1=1}^{m(2)-h_2-\cdots-h_{u-1}}[m(1) - h_1 - \cdots - h_{u-1}]\right\},$$

where

$$w = i_s + j_s - 1,\quad u = j_s - 1,\quad m(j_t + v_t - 1) = i_s + j_s - i_t - j_t - v_t + 1$$
$$t = 0, 1, \cdots, s-1,\qquad 1 \le v_t \le j_{t+1} - j_t,\qquad i_0 = j_0 - 1 = 0.$$

Proof. It is sufficient to prove the theorem for the expression

$$\Pr\left\{\max[x(n+1-j_k) + x(i_k);\ 1 \le k \le s] < 2\phi\right\}$$

since any probability expression of the form $\Pr\{\beta\text{th largest of } [\ ] < 2\phi\}$ can be expressed as a specified constant plus a sum of probabilities of the form $\Pr\{\max[\ ] < 2\phi\}$ multiplied by specified constants, where in each case the terms in the [ ] are a subset of the s terms: $x(n+1-j_k) + x(i_k)$, $(1 \le k \le s)$.

Let the integer n have the value n_o. Then it can be verified that

$$(6)\qquad \Pr\left\{\max[x(n_o+1-j_k) + x(i_k);\ 1 \le k \le s] < 2\phi\right\}$$
$$= \Pr\left[\max\left\{2x(n_o-j_s),\ x[n_o+1-W] + x[n_o+1-W-m(W)];\ 1 \le W \le j_s\right\} < 2\phi\right],$$

where

$$m(j_t + v_t - 1) = n_o + 2 - i_t - j_t - v_t,\qquad m(j_s) = n_o - i_s - j_s \ge 1,$$
$$t = 0, 1, \cdots, s-1,\quad 1 \le v_t \le j_{t+1} - j_t,\quad i_o = j_o - 1 = 0,$$

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by the use of Theorem 4 of [2]. By the proof of Theorem 5 of [2], the value of the second term in (6) equals

$$\Pr\left[\max\left\{2x(n_o-j_s),\ x[n_o+2-W]+x[n_o+1-W-m(W)];\ 1\leq W\leq j_s+1\right\}<2\phi\right]$$

if $m(j_s+1)=1$ and the expression is based on n_o+1 rather than n_o observations (the values of the m's are the same as in (6)). The value of this expression, however, can be shown to equal the value of

$$\Pr\left[\max\left\{2x(n_o+1-j_s),\ x[n_o+2-W]+x[n_o+2-W-m(W)];\ 1\leq W\leq j_s\right\}<2\phi\right],$$

which by (6) equals the value of

$$\Pr\left\{\max[x(n_o+2-j_k)+x(i_k);\ 1\leq k\leq s]<2\phi\right\}$$

if $n=n_o+1$ for this expression. Thus, by induction, the value of

$$\Pr\left\{\max[x(n+1-j_k)+x(i_k);\ 1\leq k\leq s]<2\phi\right\}$$

is the same for all sample sizes $n\geq i_s+j_s$. An analysis similar to that used in the proof of Theorem 5 of [2] shows that this also holds for $n=i_s+j_s-1$. Equation (5) was obtained by taking $n=w=i_s+j_s-1$, the m's as given by (6) with this value of n, and substituting into Theorem 4 of [2].

Another basic result is that, if the observations are from continuous symmetrical populations with common median ϕ, the value of

$$\Pr\left\{\min[x(n+1-i_k)+x(j_k);\ 1\leq k\leq s]>2x(W_\alpha)\right\}$$
$$=\Pr\left\{\max[x(n+1-j_k)+x(i_k);\ 1\leq k\leq s]<2x(n+1-W_\alpha)\right\}$$

is always less than or equal to 2α. This is a particular application of the theorem

<u>Theorem 2</u>. <u>Consider</u> n <u>independent observations from continuous symmetrical populations with common median</u> ϕ. <u>Then, for any integer</u> W,

$$\Pr\left\{\max[x(n+1-j_k)+x(i_k);\ 1\leq k\leq s]<2x(W)\right\}$$
$$\leq\Pr\left\{\max[x(n+1-j_k)+x(i_k)]<2\phi\right\}+\Pr\left\{x(W)>\phi\right\}$$
$$-\Pr\left\{\max[x(n+1-j_k)+x(i_k)]<2\phi,\ x(W)>\phi\right\}.$$

<u>Proof</u>.

$$\Pr\left\{\max[\ ]<2x(W)\right\}$$
$$=\Pr\left\{\max[\ ]<2\phi,\ x(W)>\phi\right\}+\Pr\left\{\max[\ ]<2\phi,\ x(W)<\phi,\ \max[\ ]<2x(W)\right\}$$
$$+\Pr\left\{\max[\ ]>2\phi,\ x(W)>\phi,\ \max[\ ]<2x(W)\right\}$$
$$\leq\Pr\left\{\max[\ ]<2\phi,\ x(W)>\phi\right\}+\Pr\left\{\max[\ ]<2\phi,\ x(W)<\phi\right\}$$
$$+\Pr\left\{\max[\ ]>2\phi,\ x(W)>\phi\right\}$$
$$=\Pr\left\{\max[\ ]<2\phi\right\}+\Pr\left\{x(W)>\phi\right\}-\Pr\left\{\max[\ ]<2\phi,\ x(W)>\phi\right\}.$$

If the n independent observations satisfy conditions (A) in addition to being from continuous symmetrical populations with a common median value, the significance level of tests

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(1) and (3) tends to α as $n \to \infty$. This follows from symmetry considerations and

Theorem 3. Consider n independent observations which satisfy conditions (A) and are from continuous symmetrical populations with a common median value. Then

$$\lim_{n\to\infty} \Pr\left\{\min[x(n+1-i_k) + x(j_k);\ 1 \leq k \leq s] > 2x(W_\alpha)\right\} = \alpha.$$

Proof. Let

$$Y = \min\left[x(n+1-i_k) + x(j_k);\ 1 \leq k \leq s\right]$$

and consider the case where

$$\lim_{n\to\infty} \sigma[x(W_\alpha)]/\sigma(Y) = 0.$$

Since the populations are continuous, $\sigma(Y) > 0$ and

$$\Pr[Y > 2x(W_\alpha)] = \Pr[Y - 2\phi > 2x(W_\alpha) - 2\phi]$$
$$= \Pr\left\{[Y - 2\phi]/\sigma(Y) > 2[x(W_\alpha) - \phi]/\sigma(Y)\right\}.$$

Let

$$Z = 2[x(W_\alpha) - \phi]/\sigma(Y).$$

Then, from (i) of conditions (A),

$$\Pr[Y > 2x(W_\alpha)] = \int_{-\infty}^{\infty} \Pr\left\{[Y - 2\phi]/\sigma(Y) > a\right\} dF_z(a) + \beta(n),$$

where F_z is the cdf of Z and $\lim_{n\to\infty} \beta(n) = 0$.

Let b be any positive number. From $\lim_{n\to\infty} \sigma(Z) = 0$, (ii) of conditions (A), and the definition of $x(W_\alpha)$, the mean of Z exists for all values of n and tends to zero as $n \to \infty$. Then, by Tchebycheff's Inequality, it can be shown that

$$\int_{-b}^{b} dF_z(a) = 1 - \gamma(n),$$

where $\lim_{n\to\infty} \gamma(n) = 0$.

From (iii) of conditions (A)

$$\lim_{n\to\infty} \Pr\left\{[Y - 2\phi]/\sigma(Y) > -b\right\} = \lim_{n\to\infty} \Pr\left\{[Y - 2\phi]/\sigma(Y) > b\right\} + \delta(b),$$

where $\lim_{b\to 0} \delta(b) = 0$.

Using the above relations, letting $n \to \infty$ first and then $b \to 0$, it follows from Theorem 1 that

$$\lim_{n\to\infty} \Pr[Y > 2x(W_\alpha)] = \Pr\left\{[Y - 2\phi]/\sigma(Y) > 0\right\} = \alpha.$$

A similar type proof shows that this limiting relation also holds when

$$\lim_{n\to\infty} \sigma[x(W_\alpha)]/\sigma(Y) = \infty.$$

Finally consider properties of the power functions of tests (1) and (3) for the special situation outlined in sections 1 and 2. The properties stated in the preceding two sections

follow from

Theorem 4. Let $x(n \quad 1 - r), \cdots, x(n)$ be from continuous symmetrical populations with common median θ, the remaining order statistics from continuous symmetrical populations with common median ϕ, and each population have the property that the distribution of $x - \phi$ is independent of ϕ, where x is an observation from the population and ϕ is the median of the population. Also let

$$P_1(\Phi) = \Pr\left\{\min[x(n+1-i_k) + x(j_k);\ 1 \le k \le s \le r] > 2x(W_\alpha) \,\middle|\, \theta - \phi = \Phi\right\},$$

where the conditions for test (1) are satisfied, and

$$P_3(\Phi) = \Pr\left\{\max[x(n+1-j_k) + x(i_k);\ 1 \le k \ge s \le r] < 2x(n+1-W_\alpha) \,\middle|\, \theta - \phi = \Phi\right\},$$

where the conditions for test (3) are satisfied. Then

$$\lim_{\Phi \to -\infty} P_1(\Phi) = 0, \qquad \lim_{\Phi \to \infty} P_1(\Phi) = 1,$$

$$\lim_{\Phi \to -\infty} P_3(\Phi) = 1, \qquad \lim_{\Phi \to \infty} P_3(\Phi) = 0,$$

$P_1(\Phi)$ is a monotonically increasing function of Φ for $\Phi < 0$, and $P_3(\Phi)$ is a monotonically decreasing function of Φ for $\Phi < 0$.

Proof. It is sufficient to prove this theorem for the power function of test (3). The results for $P_1(\Phi)$ can be obtained from symmetry considerations and obvious modifications of the proof for $P_3(\Phi)$.

First consider $P_3(\Phi)$ for the case where $\Phi \le 0$. Let a new set of observations be formed from the given set by subtracting the median value of the corresponding population from each observation. Let $y(1), \cdots, y(n)$ be the values of the set of modified observations arranged in increasing order of magnitude. Since $\Phi \le 0$, $\theta \le \phi$ and

$$y(t) = \begin{cases} x(t) - \phi, & 1 \le t \le n - r \\ x(t) - \theta, & n - r + 1 \le t \le n. \end{cases}$$

Thus

$$P_3(\Phi) = \Pr\left\{\max[y(n+1-j_k) + y(i_k);\ 1 \le k \le s \le r] - 2y(n+1-W_\alpha) < -\Phi\right\},$$

whence it follows that $P_3(\Phi)$ is a monotonically decreasing function of Φ for $\Phi \le 0$ and that $\lim_{\Phi \to -\infty} P_3(\Phi) = 1$.

Now consider the case where $\Phi > 0$. Again form the set of modified observations and let $y(1), \cdots, y(n)$ be the values of these observations arranged in increasing order of magnitude. Then it is easily seen that

$$P_3(\Phi) \le \Pr[y(1) - y(n) < -\tfrac{1}{2}\Phi]$$

so that $\lim_{\Phi \to \infty} P_3(\Phi) = 0$.

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